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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

IV.

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4. EARLY DISCUSSIONS OF LIMITS: GREGORY ST. VINCENT, GALILEO, HOBBES.

Limits in the Fifteenth and Sixteenth Centuries. With the fifteenth century new mathematical ideas appear. These germs are found in Greek philosophy, but they failed to develop during the dark centuries. In the fifteenth century the German cardinal, Nicolaus Cusanus (1401–1465), considered variability without being able to apply it successfully; he advanced the notion of a limit, though unable to pass correctly to the limit; he entertained the notion of infinitesimals but was not able to use them in an infinitesimal calculus.[1] He held that rules developed for the finite lose their validity for the infinite—a statement which later thinkers have not always heeded sufficiently. A point moving with infinite velocity in a circle is each moment in every position on the circle; hence it is at rest.

During the century, or century and a half, after Cusanus, concepts of limits and processes involving the passing to the limit begin to appear in different parts of Europe, like flowers on a field in early spring. Perhaps first in time, in the development of ideas considered by Cusanus, is Giovanno B. Benedetti, a distinguished forerunner of Galileo, who brought out a publication in 1585 at Turin, Italy. As early as 1586, and again in 1608, Simon Stevin at Leyden exhibited the process of passing to the limit.[2] In 1604 the Italian mathematician, Luc Valerio, published at Rome a treatise, *De centrogravitatis*, which contains a remarkable approach to the modern idea of limits.[3] In Galileo's celebrated discourses on mechanics and falling bodies (1638) there are frequent instances of limits. In the Netherlands again, Gregory St. Vincent, whose researches have, until recently, hardly received the recognition they deserve, was familiar with the writings of Luc Valerio, and himself contributed toward laying the foundations for the infinitesimal calculus. Similar studies bearing on the concept of a limit are due to Andreas Tacquet of Antwerp, and to John Wallis in his *Arithmetica infinitorum*, 1655, who were both familiar with the *Opus geometricum* of Gregory St. Vincent.[1]

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[1] K. Lasswitz, *Geschichte der Atomistik*, Hamburg und Leipzig, 1. Bd., 1890, pp. 283, 284, 287, See also Max Simon, "Cusanus als Mathematiker," *Festschr. H. Weber*, Leipzig und Berlin, 1912. pp. 298–337.

[2] H. Bosmans, "Sur quelques exemples de la méthode des limites chez Simon Stevin," *Annales de la société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule.

[3] H. Bosmans, "Les démonstrations par l'analyse infinitésimale chez Luc Valerio," *Annales de la Société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule; C. R. Wallner, "Ueber die Entstehung des Grenzbegriffes," *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, p. 250,

We proceed now to a special mention of discussions of Zeno's arguments. Benedetti, whom we mentioned above, held that the flying arrow, thought of at a point in its path, does not cover a finite distance, but it differs from an arrow at rest by possessing the attribute of velocity which persists even in an infinitesimal time and space.[2] Direct reference to Zeno in a manner which exhibits reckless following of the great dialectician is found in Giuseppe Biancani of Bologna who about 1615 sought to establish the incommensurability of two lines by the consideration that a supposed common measure could not be applied to either line, because the measure must first be applied to half of it, and before that to half of that half, and so on to infinity, which is as impossible an operation as Zeno's "Dichotomy."[3]

Speculations of Galileo. Far more successful than earlier writers in the application of infinitesimals were Kepler and Cavalieri, but more important to us at present are the speculations of Galileo. Galileo approached the problem of infinite aggregates with a keenness of vision and an originality which was not equalled before the time of Dedekind and Georg Cantor. Galileo's dialogues on mechanics, *Discorsi e Dimostrazioni matematiche*, 1638, opens the "first day" with a discussion of divisibility and continuity of matter and space.[4] Salviati, who in general represents the author's own ideas, says, [5]"the infinite is inconceivable to us, as is the last indivisible." Simplicio, who in these dialogues is the spokesman of Aristotelian scholastic philosophy, remarks that "the infinity of points on a longer line must be greater than the infinity of points on a shorter one." Then come the remarkable words of Salviati:

> "These difficulties arise because we with our finite minds discuss the infinite, attributing to the latter properties derived from the finite and limited. This, however, is not justifiable; for the attributes great, small and equal are not applicable to the infinite, since one cannot speak of greater, smaller, or equal infinities. . . . If now I ask how many squares are there, one can answer with truth, just as many as there are roots; for every square has a root,[6] every root has a square, no square has more than one root, no root more than one square. . . . I see no escape, except to say: the totality of numbers is infinite, the totality of squares is infinite, the totality of roots is infinite; the multitude of squares is not less than the multitude of numbers, neither is

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[1] C. R. Wallner, *loc. cit.*, p. 257.

[2] K. Lasswitz, *op. cit.*, Vol. II, p. 17.

[3] J. C. Heilbronner, *Historia matheseos universæ*, Lipsiæ, 1742, p. 175.

[4] See a German translation in *Ostwald's Klassiker*, No. 11, pp. 24–37, also No. 24, p. 17; an English translation of the parts bearing on aggregates is given by E. Kasner in *Bulletin Am. Math. Soc.*, Vol. XI, 1904–5, pp. 499–501.

[5] *Ostwald's Klassiker*, No. 11, p. 29.

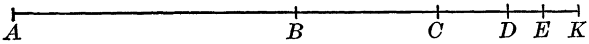
[6] Following the custom of his time, Galileo considers only one root of a positive number, namely the principal root.

the latter the greater; and, finally, the attributes equal, greater, and less are not applicable to infinite but solely to finite quantities."

We shall see that Galileo has been curiously misinterpreted by some writers, including Cauchy, as demonstrating here that an actual infinity has no existence. That there should be as many squares as there are integers altogether was taken as absurd; hence the existence of actual infinity was considered disproved. Galileo's skill in the use of the infinite in demonstrations is shown in the following passage on falling bodies:[1]

> "If the velocity were proportional to the distance through which it has fallen or is to fall, then those distances would be passed over in equal times; thus, if the velocity with which a body overcomes four yards is to be double the velocity with which the first two yards were overcome, then the times needed for these two processes would be the same; but four yards can be overcome in the same time as two yards only in the case of instantaneous motion; we see on the contrary that the body needs time to fall, and that it needs less time for a fall of two yards than of four yards; hence it is not true that the velocities increase proportionally to the distance fallen."

Gregory St. Vincent. The most important discussion of Zeno given at this time is that by Gregory St. Vincent, in his *Opus geometricum quadraturæ circuli et sectionum coni,* published at Antwerp in 1647, but written apparently twenty-five years earlier. It is a massive volume of 1400 pages. Influenced in his geometrical researches by the medieval scholastic concept of the continuum, according to which a line divided repeatedly is not reduced to indivisible elements as taught by the atomists, but admits of being subdivided *ad infinitum,* Gregory St. Vincent took a step different from that of Archimedes. While, in his proofs, Archimedes kept on dividing, only until a certain degree of smallness was reached, St. Vincent permitted the subdivisions to continue *ad infinitum.* Using unlimited section in geometry he introduced a geometric series that was truly an *infinite* series.[2]



This much had been accomplished by at least one writer before him,[3] but, so far as now known, he is the first to apply the infinite geometric progression to the study of the "Achilles." Taking a definite line segment AK he divides it at B in a given ratio, then he divides BK in the same ratio at C, and so on. The segments AB, BC, CD, . . . form an infinite geometric progression. The points C, D, E . . . lie, all of them, between A and K; they approach K as near as we please, but (in accordance with scholastic philosophy) never reach it. As Gregory conceives this matter, K is an obstacle, so to speak, against the further advance of the series of points A, B, C, . . . , similar to a rigid wall. "Terminus progressionis est seriei finis, ad quem nulla progressio pertinget, *licet* in infinitum continuetur; sed quovis interuallo dato proprius ad eum *accedere* poterit." By "series" is meant the segment AK, by "progressio," the segments AB, AC, ...

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[1] *Ostwald's Klassiker,* No. 24, p. 17.

[2] Gregory St. Vincent, *Opus geometricum,* T. 1, pp. 51–56, 95–97; for our knowledge of this part of the book we are dependent entirely upon C. R. Wallner's account in the *Bibliotheca mathematica,* 3. F., Vol. IV, 1903, pp. 251–255.

[3] See H. Wieleitner in *Bibliotheca mathematica,* 3. F., Vol. 14, 1914, pp. 150–168.

Gregory states his conclusion thus: "Dico magnitudinem AK aequalem esse toti progressioni magnitudinum continue proportionalium, rationis AB ad BC in infinitum continuatae; siue quod idem est, rationis AB ad BC in infinitum continuatae terminum esse K." Considering the "Achilles" in this connection, he associates this paradox on motion for the first time definitely with the summation of an infinite series. Moreover, Gregory St. Vincent is the first writer known to us who states the exact time and place of overtaking the tortoise. So far as we are able to ascertain, Gregory was not troubled, in explaining the "Achilles," by the fact that in his theory, the variable does not *reach* its limit. Nor, apparently, did this matter trouble his readers. His mode of solving the problem appealed to many. We shall see that Leibniz makes special reference to it. Over a century after Gregory's publication, Saverien refers in his dictionary[1] to the "Achilles," "dont Gregoire de Saint Vincent a fait voir la fausseté." Formey gave Gregory St. Vincent's explanation in the article "Mouvement" in Diderot's *Encyclopédie* (1754), later reprinted in the *Encyclopédie méthodique,* and in 1800 translated at Padova into the Italian language. The definition of a limit as given in the *Encyclopédie méthodique* does not allow the variable to surpass its limit but places no obstacle in the way of its reaching its limit.

Descartes, De Morgan and Others. Descartes at one time discussed the "Achilles." His treatment is much like that of Gregory St. Vincent. It is given in a letter of July, 1646, to Clerselier.[2] He lets Achilles, or in his place a horse, be, at the start, 10 leagues behind the tortoise, but moving ten times more rapidly than the latter. The real difficulty of the paradox he does not touch, for he says:

> "L'Achille de Zenon ne sera pas difficile à soudre, si on prend garde que, si à la dixiéme partie de quelque quantité on adioute la dixiéme de cette dixiéme, qui est une centiéme, & encore la dixiéme de cette derniere, qui n'est qu'une milliesme de la premiere, & ainsi à l'infini, toutes ces dixiémes jointes ensemble, quoy qu'elles soient suposées réellement infinies, ne composent toutesfois qu' une quantité finie, sçauvoir une neusiéme de la premiere quantité . . . Et la caption est en ce qu'on imagine que cette neusiéme partie d'une lieuë est une quantité infinie, à cause qu'on la divise par son imagination en des parties infinies."

Descartes looked upon the actually infinite as mysterious, but not impossible or absurd. He seemed to accept it in the abstract, but deny it in the concrete. At this time and even earlier (see the foregoing extracts from Galileo) there was talk about the finitude of the human mind and its consequent inability to conceive the infinite. This was ridiculed by De Morgan. He claimed that if the human mind is limited, we tacitly postulate the "unknowable"; moreover, even if the human mind were finite, there is no more reason against its conceiving the infinite than there is for a mind to be blue in order to conceive of a pair of blue eyes. Or, as De Morgan puts it in another place, the argument amounts to this, "who drives fat oxen should himself be fat." From Descartes to Hamilton, says De Morgan,[1] this doctrine is accepted by many minds. But its genesis is found, as we have stated, long before Descartes.

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[1] Saverien, *Dictionnaire universel de mathèmatique et de physique,* Paris, 1753, Art. "Mouvement."

[2] *Oeuvres de Descartes* par Charles Adam et Paul Tannery, T. IV, pp. 445–447.

A wholly different, but no more satisfactory explanation of "Achilles" comes from another Frenchman of that time, Pierre Gassendi, the physicist. In his view Zeno's proofs need no refutation, if with Epicurus one assumes not points but atoms. A difficulty seems to arise from differences in velocity of motion, for in the same time that a body moves over the physically indivisible, the more rapid body must travel over several indivisibles. In his opinion this difficulty may perhaps be overcome by conceiving motion as discontinuous, and slower motion as a mixture of rest and motion. To the senses motion would still seem continuous.[2] To those who experienced difficulties in accepting the existence of indivisible atoms, the capuchin, Casimir of Toulouse, offers an easy solution by reminding that angels had extension, yet were physically indivisible.[3]

It is worthy of note that John Dee, the famous astrologer who wrote an elaborate mathematical preface to Billingsley's edition of Euclid (1570), departs from the contention that two lines containing the same number of parts must be of equal length. He says:

> "Our least Magnitudes can be divided into so many partes as the greatest. As, a Line of an inch long (with vs) may be divided into as many partes, as may the diameter of the whole world, from East to West: or any way extended."

Discussion of Thomas Hobbes. The earliest British writer, after Duns Scotus, to take up explicitly Zeno's arguments is the philosopher, Thomas Hobbes (1588–1679). In 1655 he wrote:[4]

> ". . . the force of that famous argument of Zeno against motion, consisted in this proposition, *whatsoever may be divided into parts, infinite in number, the same is infinite;* which he without doubt, thought to be true, yet nevertheless is false. For to be divided into infinite parts, is nothing else but to be divided into as many parts as any man will. But it is not necessary that a line should have parts infinite in number, or be infinite, because I can divide and subdivide it as often as I please; for how many parts soever I make, yet their number is finite; because he that says parts, simply, without adding how many, does not limit any number, but leaves it to the determination of the hearer, therefore we say commonly, a line may be divided infinitely; which cannot be true in any other sense."

With Hobbes, *infinite* is synonymous with *indefinite.* He takes an agnostic attitude toward problems of infinity:

> "But when no more is said than this, *number is infinite,* it is to be understood as if it were said, this name *number* is an *indefinite* name. . . . And, therefore, that which is commonly said, that space and time may be divided infinitely, is not to be so understood, as if there might be any infinite or eternal division; but rather to be taken in this sense, *whatever is divided is divided into such parts as may again be divided.* . . . Who can commend him that demonstrates thus? 'If the world be eternal, then an infinite number of days, or other measures of time, preceded the birth of Abraham. But the birth of Abraham preceded the birth of Isaac; and therefore one

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[1] A. De Morgan, "On Infinity; and on the Sign of Equality," in Trans. of the Cambridge Philosoph. Society, Vol. XI, p. 157, Cambridge, 1871 [read May 16, 1864].

[2] Gassendi, *Opera omnia,* 1658, I, p. 300a. An. I, p. 239; Lasswitz, *op. cit.,* Vol. II, p. 150.

[3] Lasswitz, *op. cit.,* Vol. II, p. 494.

[4] *The English Works of Thomas Hobbes,* Vol. I, London, 1839, pp. 63, 64, 413. Hobbes refers to Zeno's arguments also in his Latin works. See Thomae Hobbes, *Opera philosophica,* Vol. V, Londini, 1845, pp. 207–213.

> infinite is greater than another infinite, or one eternal than another eternal; which,' he says, 'is absurd.' This demonstration is like his, who from this, that the number of even numbers is infinite, would conclude that there are as many even numbers as there are numbers simply, that is to say, the even numbers are as many as all the even and odd together. They which in this manner take away eternity from the world, do they not by the same means take away eternity from the creator of the world? . . . And the men that reason thus absurdly are not idiots, but, which makes this absurdity unpardonable, geometricians, and such as take upon them to be judges, impertinent, but severe judges of other men's demonstrations."

The reference to odd and even numbers doubtless arose from his contact with Galilean thought. While sojourning on the Continent, he had gone to see Galileo, then a prisoner. Hobbes thought he had effected the duplication of the cube and the squaring of the circle. On this matter he became involved in a heated controversy with the algebraist, John Wallis. The aged Hobbes was no match against young Wallis on mathematical questions. When the mathematical works of Wallis were being brought out, Wallis refused to allow his controversial matter against Hobbes to be incorporated in them.[1] Whether the whole is greater than a part was an issue touched upon during this dispute. Hobbes said to Wallis: "All this arguing of infinities is but the ambition of school boys." It cannot be said that Hobbes made any real contribution to a deeper understanding of the "Achilles" or any of Zeno's other arguments on motion. His objection to the dictum, "whatever may be divided into parts infinite in number, the same is infinite," is no new contribution; Aristotle had advanced that far. How Achilles caught the turtle is beyond comprehension through our sensual imagination; Hobbes nowhere explains this inability. However, he does touch upon the concept of a limit in his controversy with Wallis. Hobbes charged that some of the principles of the professors are "void of sense"; one of those principles being, "that a quantity may grow less and less eternally, so as at last to be equal to another quantity; or, which is all one, that there is a last in eternity."[2]

A GENERAL FORMULA FOR THE VALUATION OF SECURITIES.[3]

By JAMES W. GLOVER, University of Michigan.

The object of this paper is to derive a formula for the valuation of a very general type of securities. The security is redeemed in r equal installments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m installments, and the security is purchased to realize the investor a nominal rate of interest j with frequency of conversion m.

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[1] A full account of the controversy between Hobbes and Wallis is given in Croom Robertson's *Hobbes*, pp. 167–185.

[2] *The English Works of Thomas Hobbes*, Vol. 7, p. 186.

[3] Read before the Chicago Section of the American Mathematical Society, April, 1912. Those unfamiliar with the notation and functions employed in the theory of compound interest may consult *Text-Book of the Institute of Actuaries*, Part I, by Ralph Todhunter; *The Mathematical Theory of Investment*, by Ernest B. Skinner; *Bulletin of the Department of Agriculture*, No. 136, on Highway Bonds, by Laurence I. Hewes and James W. Glover.